

SINGULAR BLOCKS OF RESTRICTED  $\mathfrak{sl}_3$ 

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ABSTRACT. We compute generators and relations for the basic algebra of a non-semisimple singular block of the restricted enveloping algebra of  $\mathfrak{sl}_3$  over a field of characteristic  $p > 3$ . Working directly with the basic algebra we compute its centre and the internal degree zero part of its first Hochschild cohomology, and show its Verma modules are Koszul.

Let  $A$  be a finite-dimensional algebra over an algebraically closed field  $k$ . The basic algebra of  $A$  is a  $k$ -algebra all of whose simple modules are one-dimensional which is Morita equivalent to  $A$ ; it can be thought of as the smallest algebra with the same representation theory as  $A$ . The small size of the basic algebra tends to make calculations considerably easier.

This paper deals with restricted enveloping algebras of classical Lie algebras. While the basic algebras for non-semisimple blocks of restricted  $\mathfrak{sl}_2$  have long been known (and are reviewed in Section 6), to the best of my knowledge no other nontrivial examples have been computed, although some endomorphism algebras of projectives are constructed in [AJS94, Chapter 19]. Here we find generators and relations for the basic algebras of non-semisimple singular blocks of the restricted enveloping algebra  $u$  of  $\mathfrak{sl}_3(k)$  over an algebraically closed field of characteristic  $p > 3$ , and some of their properties. The simple modules of  $u$  are labelled by their highest weights; singular here means that the block has a simple module whose highest weight  $\lambda$  satisfies  $\langle \lambda + \rho, \alpha \rangle = 0 \pmod p$  for some root  $\alpha$ , where  $\rho$  is the half-sum of the positive roots for  $\mathfrak{sl}_3(k)$ .

The method is as follows. Thanks to work of Simon Riche [Ric10] we know that each block of  $u$  admits a  $\mathbb{Z}_{\geq 0}$ -grading with respect to which it is a Koszul algebra. We show in Section 2 that the Verma modules in our block are graded modules with respect to this grading. This allows us to use a version of the Brauer-type reciprocity of [HN91] to find the graded structure of the projectives in our block, and hence the Hilbert series for the basic algebra and of its Koszul dual.

In Section 4 we show, using the Frobenius homomorphism for the hyperalgebra of  $\mathfrak{sl}_3(k)$ , that the basic algebra admits a non-trivial action of  $\mathfrak{sl}_3(k)$  by derivations. We determine this action on the degree one part of the basic algebra, which allows us to calculate its relations in Theorem 5.1. In Section 6 we use this presentation of the basic algebra to calculate its centre and the internal degree zero part of its first Hochschild cohomology. Finally in Section 7 we identify the modules for the basic algebra that correspond to the Verma modules in our block, and prove that they are Koszul.

## 1. NOTATION AND DEFINITIONS

**1.1. Algebras.** Let  $\mathbb{F}$  be a field. The Lie algebra  $\mathfrak{sl}_3(\mathbb{F})$  is generated by elements  $E_i, F_i, H_i$  for  $i = 1, 2$  subject to the relations

$$\begin{array}{lll} \text{(r1)} & [H_1, H_2] = 0 & \text{(r2)} \quad [H_i, E_i] = 2E_i \quad \text{(r3)} \quad [H_i, F_i] = -2F_i \\ \text{(r4)} & [H_i, E_j] = -E_j & \text{(r5)} \quad [H_i, F_j] = F_j \quad \text{(r6)} \quad [E_i, F_j] = 0 \\ \text{(r7)} & [E_i, [E_1, E_2]] = 0 & \text{(r8)} \quad [F_i, [F_2, F_1]] = 0 \quad \text{(r9)} \quad [E_i, F_i] = H_i \end{array}$$

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for  $i = 1, 2$  and  $j \neq i$ . We write  $E_{12}$  for  $[E_1, E_2]$  and  $F_{12}$  for  $[F_2, F_1]$ .

Let  $U_{\mathbb{Z}}$  be the Kostant  $\mathbb{Z}$ -form of the universal enveloping algebra of  $\mathfrak{sl}_3(\mathbb{C})$ : this is the subring generated by all elements of the form  $E_i^{(n)} := E_i^n/n!$  and  $F_i^{(n)} := F_i^n/n!$ . Let  $k$  be an algebraically closed field of characteristic  $p > 3$  and  $D = U_{\mathbb{Z}} \otimes_{\mathbb{Z}} k$  be the hyperalgebra or algebra of distributions [Jan03, I.7, II.1.12]. We will abuse notation by writing  $E_i^{(n)}$  for the element  $E_i^{(n)} \otimes 1$  of  $D$  and so on. The restricted enveloping algebra  $u$  of  $\mathfrak{sl}_3(k)$  is defined to be the subalgebra of  $D$  generated by the  $E_i$  and  $F_i$  for  $i = 1, 2$ , it is a Hopf algebra of dimension  $p^8$  isomorphic to the quotient of the universal enveloping algebra  $U(\mathfrak{sl}_3(k))$  by the two-sided ideal generated by  $E_i^p, F_i^p, E_{12}^p, F_{12}^p, H_i^p - H_i$  for  $i = 1, 2$ . We write  $b^+$  for the subalgebra of  $u$  generated by the  $E_i$  and  $H_i$  for  $i = 1, 2$ , and  $b^-$  for the subalgebra generated by the  $F_i$  and  $H_i$  for  $i = 1, 2$ .

Let  $D_2$  be the subalgebra of  $D$  generated by the  $E_i^{(n)}$  and  $F_i^{(n)}$  for  $n < p^2$ . It has a basis consisting of all monomials

$$F_1^{(a_1)} F_2^{(a_2)} F_{12}^{(a_3)} \binom{H_1}{a_4} \binom{H_2}{a_5} E_1^{(a_6)} E_2^{(a_7)} E_{12}^{(a_8)}$$

where  $0 \leq a_i < p^2$  and  $\binom{H_i}{n}$  is the image in  $D_2$  of the element

$$\frac{H_i(H_i - 1) \cdots (H_i - n + 1)}{n!}$$

of  $U_{\mathbb{Z}}$ . There is a surjective algebra homomorphism  $\text{Fr} : D_2 \rightarrow u$  such that

$$\text{Fr}(E_i^{(r)}) = \begin{cases} E_i^{(r/p)} & \text{if } p \mid r \\ 0 & \text{otherwise} \end{cases}$$

and similarly for  $F_i^{(r)}$ , called the Frobenius map. The kernel of  $\text{Fr}$  is the two-sided ideal generated by the augmentation ideal of  $u$ .

**1.2. Gradings.** Let  $\Phi = \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1 + \alpha_2)\}$  be the  $\mathfrak{sl}_3$  root system with simple roots  $\alpha_1$  and  $\alpha_2$ , and let  $\varpi_1$  and  $\varpi_2$  be the fundamental weights. Let  $X$  be the weight lattice, the free abelian group on the  $\varpi_i$ , which we identify with  $\mathbb{Z}^2$  by  $\varpi_1 = (1, 0)$  and  $\varpi_2 = (0, 1)$ . Then  $U_{\mathbb{Z}}$ ,  $D$  and  $u$  are  $X$ -graded with  $E_i$  in degree  $\alpha_i$  and  $F_i$  in degree  $-\alpha_i$ . We refer to this as the weight grading and write  $|x| = \alpha$  if  $x$  is a homogeneous element of degree  $\alpha \in X$ . Let  $W$  be the Weyl group of  $\Phi$ , generated by reflections  $s_i$  in the hyperplane perpendicular to  $\alpha_i$ , so  $W$  acts by automorphisms on the restricted Lie algebra  $\mathfrak{sl}_3(k)$  and hence on  $u$ . Thinking of  $\mathfrak{sl}_3(k)$  as traceless  $3 \times 3$  matrices over  $k$ , this action comes from conjugation by permutation matrices.

By [Ric10],  $u$  admits a  $\mathbb{Z}_{\geq 0}$ -grading with respect to which it is a Koszul algebra in the sense of [BGS96]. We write  $u_i$  for the  $i$ th graded piece, and refer to this as the  $K$ -grading on  $u$ . Koszulity implies that  $u$  is generated as an algebra by  $u_0$  and  $u_1$ , and that  $u_{>0} = \bigoplus_{i>0} u_i$  is the Jacobson radical  $J$  of  $u$ .

**1.3. Modules.** All modules considered in this paper will be left-modules unless otherwise stated.

If  $M = \bigoplus_{\alpha \in X} M_{\alpha}$  is a weight-graded  $u$ -module and  $w \in W$  we write  ${}^w M$  for the weight-graded  $u$ -module with  $({}^w M)_{\alpha} = M_{w(\alpha)}$  for any  $\alpha \in X$  and action  $x \cdot m = w(x)m$  for  $x \in u$  and  $m \in M$ . If  $i, j \in \mathbb{Z}$  then  $M[i, j]$  is the graded module which has the same underlying vector space and action as  $M$ , but with grading determined by  $M[i, j]_{\alpha} = M_{\alpha - (i, j)}$  for any  $\alpha \in X$ . We write  $\text{Hom}_u^{\text{wt}}$  and  $\text{Ext}_u^{\text{wt}}$  for the Hom and Ext functors in the category of finite-dimensional weight-graded  $u$ -modules, so an element  $f$  of  $\text{Hom}_u^{\text{wt}}(M, N)$  sends  $M_{\alpha}$  to  $N_{\alpha}$  for all  $\alpha \in X$ . If

$\text{Hom}_u$  denotes the hom-functor in the ungraded category then for finite-dimensional weight-graded  $u$ -modules  $M$  and  $N$ ,

$$(1) \quad \text{Hom}_u(M, N) = \bigoplus_{\alpha \in X} \text{Hom}_u^{\text{wt}}(M, N[\alpha])$$

with a similar result for  $\text{Ext}$ .

If  $M = \bigoplus_{i \in \mathbb{Z}} M_i$  is a  $K$ -graded module, we write  $M[i]$  for the  $K$ -graded module with the same underlying vector space and action as  $M$ , but with grading determined by  $M[i]_j = M_{i+j}$ . We write  $\text{Hom}_u^K$  and  $\text{Ext}_u^K$  for the Hom and Ext functors in the category of finite-dimensional  $K$ -graded modules. Again, finite-dimensionality implies

$$(2) \quad \text{Hom}_u(M, N) = \bigoplus_{i \in \mathbb{Z}} \text{Hom}_u^K(M, N[i])$$

for graded modules  $M, N$ , with a similar result for  $\text{Ext}$ . We say that a  $u$ -module  $M$  is  **$K$ -gradable** if it admits a decomposition  $M = \bigoplus_{i \in \mathbb{Z}} M_i$  as vector spaces such that  $u_i M_j \subseteq M_{i+j}$  for all  $i$  and  $j$ . The simple  $u$ -modules are  $K$ -gradable since  $u_{>0}$  acts as zero on any simple if  $i > 0$ , and the projective  $u$  modules are  $K$ -gradable [GG82].

## 2. VERMA MODULES

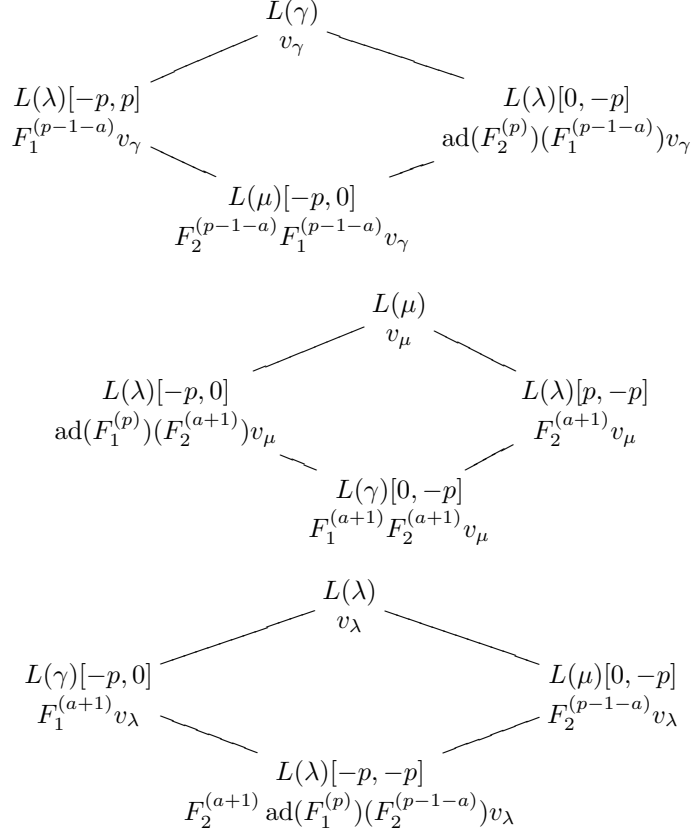
**2.1. Structure of Verma modules.** Let  $0 \leq t_1, t_2 < p$  be integers and let  $\alpha = (t_1, t_2) \in X$ . Write  $k_\alpha$  for the one-dimensional  $b^+$ -module on which  $E_i$  acts are zero and  $H_i$  as the image in  $k$  of  $t_i$ . The Verma module  $M(\alpha)$  is defined to be the induced module  $u \otimes_{b^+} k_\alpha$ , it has a unique simple quotient module written  $L(\alpha)$ . We regard  $k_\alpha$  as a weight-graded  $b^+$ -module concentrated in degree  $\alpha$ ; this induces a weight-grading on the Verma modules since  $b^+$  is a weight-graded subalgebra of  $u$ . The simple modules  $L(\alpha)$  are also weight-graded. An element  $m$  of a weight-graded module  $M$  is called a highest weight vector if it is an eigenvector for the action of the  $H_i$  and  $E_1 m = E_2 m = 0$ ; each  $M(\alpha)$  is generated by a highest weight vector  $v_\alpha = 1 \otimes 1$ , and it has a universal property that implies it surjects onto any module generated by a highest weight vector with weight  $\alpha$ . The dual Verma module  $M^-(\alpha)$  is  $M(\alpha)^* = \text{Hom}_k(M(\alpha), k)$  as a vector space, with weight grading  $M^-(\alpha)_\beta = (M(\alpha)_\beta)^*$  and action  $(x \cdot f)(m) = f(t(x)m)$  for  $m \in M(\alpha), x \in u$  and where  $t$  is the antigraded antiautomorphism of  $u$  such that  $t(E_i) = F_i$  and  $t(F_i) = E_i$  for  $i = 1, 2$  [HN91, Remark after Theorem 5.1]. In our case,  $M^-(\alpha)$  is isomorphic to the  $Z^-(\alpha)$  of [HN91] as explained in the proof of their Theorem 5.1. Since the simple  $u$ -modules are self-dual,  $M^-(\lambda)$  has simple socle  $L(\lambda)$ .

For each  $0 \leq a \leq p-2$  there is a non-semisimple singular block of  $u$  containing the simple modules  $L(p-1, a), L(a, p-2-a)$  and  $L(p-2-a, p-1)$ . From now on we fix such an  $a$  and write

$$\mu = (p-1, a), \quad \lambda = (a, p-2-a), \quad \gamma = (p-2-a, p-1).$$

The submodule lattices for the Verma modules corresponding to these weights were determined in [Xi99, Theorems 2.4, 2.5, 2.6] and [Irv86, Theorems 3.3, 3.4]. The following diagrams display these submodule lattices; beneath each composition factor  $L$  is an element of the Verma module which is a highest weight vector for  $L$

modulo the sum of the submodules below it in the lattice.



The weight-graded composition structure of the dual Verma modules follows immediately from these.

## 2.2. Verma modules are $K$ -gradable.

**Lemma 2.1.**  $\text{Ext}_u^1(L(\gamma), L(\mu)) = \text{Ext}_u^1(L(\mu), L(\gamma)) = \text{Ext}_u^1(L(\lambda), L(\lambda)) = 0$ .

*Proof.* Consider the first of these Ext groups. Using the decomposition (1) it is enough to prove that each of the weight-graded Ext groups  $\text{Ext}_u^{\text{wt}, 1}(L(\gamma), L(\mu)[\beta])$  for  $\beta \in X$  are zero. Let  $\text{supp } L(\alpha)$  denote the support of  $L(\alpha)$  in  $X$ , a convex set. If

$$0 \rightarrow L(\mu)[\beta] \rightarrow M \rightarrow L(\gamma) \rightarrow 0$$

is a non-split exact sequence of graded modules, there is a vertex of  $\text{supp } L(\gamma)$  lying outside or on the boundary of  $\beta + \text{supp } L(\mu)$ . Since  ${}^w L(\alpha) \cong L(\alpha)$  for any  $w \in W$  and any  $\alpha$ , by twisting this extension with some Weyl group element if necessary we may assume that an element of  $M$  whose weight corresponds to this vertex is a highest weight vector generating  $M$ . Thus  $M$  is a quotient of  $M(\gamma)$ , which is impossible by the structure of  $M(\gamma)$  given above. The other Ext groups follow similarly.  $\square$

**Proposition 2.2.** *Let  $\alpha \in \{\lambda, \gamma, \mu\}$ . Then  $M(\alpha)$  and  $M^-(\alpha)$  are  $K$ -gradable.*

*Proof.* We give the proof for  $M(\alpha)$ , the proof for  $M^-(\alpha)$  being analogous. The projective  $P(\alpha)$  is  $K$ -graded, generated in degree zero. Choose a surjection  $F : P(\alpha) \rightarrow M(\alpha)$  and let  $M(\alpha)_0 = F(P(\alpha)_0)$ , a  $u_0$ -submodule of  $M(\alpha)$ . Let  $M(\alpha)_2 = \text{soc}_u(M(\alpha))$ .

The quotient map  $\text{rad } M(\alpha) \rightarrow \text{rad } M(\alpha)/\text{soc } M(\alpha)$  is split as a map of  $u_0$ -modules because  $u_0$  is semisimple; let  $M(\alpha)_1 \subset M(\alpha)$  be the image of a splitting map. As  $u_0$ -modules,

$$M(\alpha)|_{u_0} = M(\alpha)_0 \oplus M(\alpha)_1 \oplus M(\alpha)_2$$

and  $M(\alpha)_1$  is isomorphic as a  $u_0$ -module to the restriction of the two middle composition factors of  $M(\alpha)$  to  $u_0$ . We claim that this is a  $K$ -grading on  $M(\alpha)$ . Since we had a decomposition of  $u_0$ -modules, the only thing we need check is that  $u_1 M(\alpha)_0 \subseteq M(\alpha)_1$ .

Let  $l_0 \in M(\alpha)_0$ ,  $x \in u_1$  and suppose  $xl_0 = l_1 + l_2$  for  $l_i \in M(\alpha)_i$ . We will show  $l_2 = 0$ . Suppose not, let  $\text{soc } M(\alpha) \cong L(\pi)$ , and write  $u_0 = \bigoplus_{\beta} M_{\beta}$  where  $M_{\beta}$  is a matrix algebra over  $k$  corresponding to the simple module  $L(\beta)$ . Pick  $p \in M_{\pi}$  such that  $pl_2 = l_2$ . By Lemma 2.1,  $JP(\alpha)/J^2P(\alpha)$  has no  $L(\pi)$  summands, so  $M_{\pi}$  acts as zero on  $P(\alpha)_1 = u_1 P(\alpha)_0$  and on the  $u_0$ -module  $M(\alpha)_1$ . Let  $\hat{l}_0 \in P(\alpha)_0$  be such that  $F(\hat{l}_0) = l_0$ . Then  $F(px\hat{l}_0) = l_2 \neq 0$  so  $p \cdot x\hat{l}_0 \neq 0$ , contradicting that  $M_{\pi}$  acts as zero on  $u_1 P(\alpha)_0$ .  $\square$

Henceforth  $M(\alpha)$  will have the  $K$ -grading constructed above, so that its top composition factor is in degree zero, and  $M^-(\alpha)$  will be given the  $K$ -grading so that its socle  $L(\alpha)$  is in degree zero and its head in degree  $-2$ . Thus the multiplicity of  $L(\alpha)[i]$  as a composition factor of  $M(\beta)$  equals the multiplicity of  $L(\alpha)[-i]$  as a composition factor of  $M^-(\beta)$ .

### 3. PROJECTIVE MODULES AND HILBERT SERIES

In this section we study the structure of the indecomposable projective modules in our block.

**3.1.  $K$ -graded Verma filtrations.** We will determine the composition factors of these projectives in the  $K$ -graded module category using a Brauer-type reciprocity result. The results of [HN91] cannot be used directly since  $b$  is not a  $K$ -graded subalgebra of  $u$ , however the proofs of the reciprocity results follow the arguments of [HN91] closely.

**Definition 3.1.** A  $K$ -graded module  $M$  is said to have a  $K$ -graded Verma filtration if there is a sequence  $0 = M_0 \subset M_1 \subset \cdots \subset M_s = M$  of  $K$ -graded submodules with each quotient  $M_r/M_{r-1}$  isomorphic in the  $K$ -graded category to  $M(\alpha_r)[n_r]$  for some  $\alpha_r \in X$  and some  $n_r \in \mathbb{Z}$ .

Write  $[M : M(\alpha_r)[n_r]]$  for the number of  $M_r/M_{r-1}$  which are isomorphic as  $K$ -graded modules to  $M(\alpha_r)[n_r]$  (it is not clear yet that this is independent of the filtration chosen), and  $[M : L(\alpha)[n]]$  for the multiplicity of  $L(\alpha)[n]$  as a composition factor of  $M$  in the category of  $K$ -graded modules. Let  $P(\alpha)$  be a projective cover of  $L(\alpha)$ .

**Lemma 3.2.** *Let  $\alpha \in \{\lambda, \mu, \gamma\}$ . Then  $P(\alpha)$  has a  $K$ -graded Verma filtration.*

*Proof.* More generally, suppose that  $M$  is any  $u$ -module in our block which is projective on restriction to  $b^-$ . By [HN91, Theorem 4.4] there is an ungraded injection  $\iota : M(\beta) \rightarrow M$  for some  $\beta \in \{\lambda, \mu, \gamma\}$ . We may decompose  $\iota$  as a sum of  $K$ -graded homomorphisms  $\iota_j$ . A homomorphism  $M(\beta) \rightarrow M$  is injective if and only if it is non-zero on the simple socle of  $M(\beta)$ , and not all the  $\iota_j$  can be zero on the socle since  $\iota$  isn't. Thus one of the  $\iota_j$  is a  $K$ -graded injective module homomorphism  $M(\beta) \rightarrow M$ . The result follows by induction on the dimension of  $M$ .  $\square$

**Proposition 3.3.** *Let  $\alpha, \beta \in \{\lambda, \gamma, \mu\}$  and  $i \in \mathbb{Z}$ . Then*

$$[P(\alpha) : M(\beta)[i]] = [M(\beta)[-i] : L(\alpha)].$$

*Proof.* We first show for any  $K$ -graded module  $M$  with a  $K$ -graded Verma filtration that

$$[M : M(\beta)[i]] = \dim \operatorname{Hom}_u^K(M, M^-(\beta)[i]).$$

The proof is by induction on the length of the Verma filtration. By [HN91, Lemma 4.2],  $\operatorname{Ext}_u^n(M(\alpha), M^-(\beta))$  is isomorphic to  $k$  if  $n = 0$  and  $\alpha = \beta$  and is zero otherwise. Using the Ext-version of (2) it follows that if  $n \neq 0$  or  $\alpha \neq \beta$  then  $\operatorname{Ext}_u^{K,n}(M(\alpha), M^-(\beta)[i]) = 0$ , and that there is a unique  $i$  such that  $\operatorname{Hom}_u^K(M(\alpha), M^-(\beta)[i])$  is nonzero. Since  $M(\alpha)$  has simple top  $L(\alpha)$  in  $K$ -degree zero and  $M^-(\alpha)$  has simple socle  $L(\alpha)$  in  $K$ -degree zero, this is  $i = 0$ . This proves the case where the Verma filtration has length one.

Now let

$$0 \rightarrow X \rightarrow M \rightarrow M(\alpha)[r] \rightarrow 0$$

be an exact sequence of graded modules, where  $X$  has a  $K$ -graded Verma filtration. For any  $\beta, j$  we can apply  $\operatorname{Hom}_u^K(-, M^-(\beta)[j])$  and get a long exact sequence in which the Ext groups vanish by [HN91, Lemma 4.2] again. So

$$0 \rightarrow \operatorname{Hom}_u^K(M(\alpha)[r], M^-(\beta)[j]) \rightarrow \operatorname{Hom}_u^K(M, M^-(\beta)[j]) \rightarrow \operatorname{Hom}_u^K(X, M^-(\beta)[j]) \rightarrow 0$$

is exact.  $[M : M(\beta)[i]]$  is the same as  $[X : M(\beta)[i]]$  except if  $\beta = \alpha, i = r$  when it is one larger. The result follows by induction.

Take  $M = P(\alpha)$ . Then  $\dim \operatorname{Hom}_u^K(P(\alpha), M^-(\beta)[i])$  is the multiplicity of  $L(\alpha)$  as a composition factor of  $M^-(\beta)[i]$ , which is equal to the multiplicity of  $L(\alpha)$  as a composition factor of  $M(\beta)[-i]$  by the remark at the end of the previous section.  $\square$

Applying this result and the structure of the Vermas given previously gives the factors in a  $K$ -graded Verma filtration of our projectives.

Projective	$K$ -graded Verma factors
$P(\lambda)$	$M(\lambda), M(\mu)[1], M(\mu)[1], M(\gamma)[1], M(\gamma)[1], M(\lambda)[2]$
$P(\gamma)$	$M(\gamma), M(\lambda)[1], M(\mu)[2]$
$P(\mu)$	$M(\mu), M(\lambda)[1], M(\gamma)[2]$

### 3.2. Hilbert series and the basic algebra.

**Definition 3.4.** The Hilbert series of our block is the  $3 \times 3$  matrix  $P(t)$  with rows and columns labelled by  $\gamma, \lambda, \mu$  whose  $\alpha, \beta$  entry is  $\sum_i t^i \dim \operatorname{Hom}_u^K(P(\alpha)[i], P(\beta))$ .

Equivalently, the  $\alpha, \beta$  entry of  $P(t)$  is the power series in  $t$  whose coefficient of  $t^i$  is the multiplicity of  $L(\alpha)[i]$  as a composition factor of  $P(\beta)$ . The  $K$ -graded structure of the Verma modules determined in Section 2 combined with the table above immediately give:

**Theorem 3.5.** *The Hilbert series of our block is*

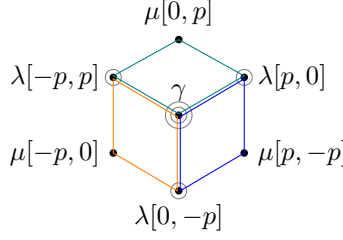
$$P(t) = \begin{pmatrix} 1 + t^2 + t^4 & 3t + 3t^3 & 3t^2 \\ 3t + 3t^3 & 1 + 10t^2 + t^4 & 3t + 3t^3 \\ 3t^2 & 3t + 3t^3 & 1 + t^2 + t^4 \end{pmatrix}.$$

The determinant of  $P(t)$  is  $(t-1)^6(t+1)^6$ . We record the following corollary on the graded structure of the projectives for future use:

**Corollary 3.6.** *The first and second  $K$ -graded pieces of the projectives in our block are as follows.*

$\alpha$	$\lambda$	$\gamma$	$\mu$
$P(\alpha)_1$	$L(\gamma)^{\oplus 3} \oplus L(\mu)^{\oplus 3}$	$L(\lambda)^{\oplus 3}$	$L(\lambda)^{\oplus 3}$
$P(\alpha)_2$	$L(\lambda)^{\oplus 10}$	$L(\gamma) \oplus L(\mu)^{\oplus 3}$	$L(\mu) \oplus L(\gamma)^{\oplus 3}$

**Definition 3.7.** The basic algebra of our block is  $E = \operatorname{End}_u(P(\mu) \oplus P(\gamma) \oplus P(\lambda))^{\operatorname{op}}$ .

FIGURE 1.  $P(\gamma)$ 

Since  $E$  is Morita equivalent [Ben98, §2.2] to a block of the Koszul algebra  $u$ , it is also Koszul with respect to the grading arising from the  $K$ -grading on the projectives [AJS94, F.3]. If we write  $e_\alpha$  for the element of  $E$  which is the identity map on  $P(\alpha)$  and zero on the other two summands, then the coefficient of  $t^i$  in the  $\alpha, \beta$  entry of the Hilbert series above equals the dimension of  $e_\beta E_i e_\alpha$ .

**Theorem 3.8.** *The Hilbert series of the Koszul dual*

$$E^\dagger \cong \text{Ext}_u^*(L(\mu) \oplus L(\gamma) \oplus L(\lambda), L(\mu) \oplus L(\gamma) \oplus L(\lambda))$$

is:

$$(t-1)^{-4}(t+1)^{-4} \begin{pmatrix} 1+4t^2+t^4 & 3t(t^2+1) & 6t^2 \\ 3t(t^2+1) & 1+4t^2+t^4 & 3t(t^2+1) \\ 6t^2 & 3t(t^2+1) & 1+4t^2+t^4 \end{pmatrix}.$$

*Proof.* By [BGS96, Lemma 2.11] the Hilbert series of  $E^\dagger$  is  $P(-t)^{-T}$ , so the result follows from Theorem 3.5.  $\square$

As power series, the diagonal entries are  $\sum_k (k+1)^3 t^{2k}$  and the off-diagonal entries are  $\frac{1}{2} \sum_k (k+1)(k+2)(2k+3)t^{2k+1}$  and  $\sum_k (k+2)(k+1)kt^{2k}$ .

**3.3. Weight-graded Verma filtrations.** Each projective has a filtration by shifts of Verma modules in the weight-graded category, and we write  $[P(\alpha) : M(\beta)[\delta]]$  for the number of factors in a weight-graded Verma filtration of  $P(\alpha)$  isomorphic in the weight-graded category to  $M(\beta)[\delta]$ . In [HN91, Theorem 5.1] the authors prove a result determining these multiplicities in a category of graded  $u$ -modules, but it is stated only for a  $\mathbb{Z}$ -grading obtained by flattening the weight grading. Their proofs go through in the weight-graded category however, or we can instead apply the  $n = 1$  case of [?, Satz 3.8], to get:

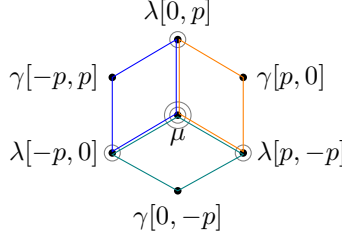
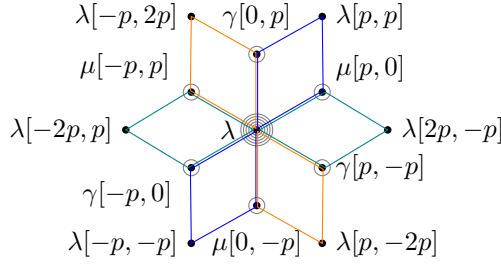
**Proposition 3.9.** *Let  $\alpha, \beta \in \{\lambda, \gamma, \mu\}$  and let  $\delta \in X$ . Then*

$$[P(\alpha) : M(\beta)[\delta]] = [M(\beta)[\delta] : L(\alpha)].$$

Again, using the structure of the Vermas given earlier gives:

Projective	Weight-graded Verma factors
$P(\gamma)$	$M(\gamma), M(\lambda)[p, 0], M(\mu)[0, p]$
$P(\lambda)$	$M(\lambda), M(\mu)[p, 0], M(\mu)[-p, p], M(\gamma)[p, -p], M(\gamma)[0, p], M(\lambda)[p, p]$
$P(\mu)$	$M(\mu), M(\lambda)[0, p], M(\gamma)[p, 0].$

This determines the composition structure of the projectives in the weight-graded category. Figures 1, 2 and 3 display these structures, where a double circle represents a composition factor of multiplicity two and so on, and we have written  $\alpha$  instead of  $L(\alpha)$ . The coloured diamonds join the composition factors of a term in the Verma filtration of  $P(\alpha)$ , blue, orange and teal diamonds are  $M(\lambda)$ ,  $M(\gamma)$  and  $M(\mu)$  respectively.

FIGURE 2.  $P(\mu)$ FIGURE 3.  $P(\lambda)$ 

## 4. ACTION BY DERIVATIONS

The aim of this section is to show that there is a non-trivial action of  $\mathfrak{sl}_3(k)$  by derivations on the basic algebra of our block: that is, a non-zero Lie homomorphism from  $\mathfrak{sl}_3(k)$  into the Lie algebra of derivations of  $E$ . Rather than work with the basic algebra as defined in the previous section, we use an equivalent definition. For  $\alpha \in \{\lambda, \gamma, \mu\}$  choose idempotents  $\varepsilon_\alpha$  belonging to a set of primitive orthogonal idempotents of  $u$  such that the image of  $\varepsilon_\alpha$  in  $u/\text{rad}(u)$  is a highest weight vector for  $L(\alpha)$ . Let  $\varepsilon = \varepsilon_\lambda + \varepsilon_\gamma + \varepsilon_\mu$ . Then  $E$  is isomorphic to  $\varepsilon u \varepsilon$  [Ben98, Lemma 1.3.3 (ii)].

For elements  $x, y$  of some algebra we write  $\text{ad}(x)(y)$  for  $xy - yx$ . The following formulas holding in  $D_2$  follow readily from [Hum78, §26.2]:

$$\begin{aligned} \text{ad}(E_i^{(p)})(F_i) &= (H_i + 1)E_i^{(p-1)} & \text{ad}(F_i^{(p)})(E_i) &= -F_i^{(p-1)}(H_i + 1) \\ \text{ad}(E_1^{(p)})(E_2) &= E_1^{(p-1)}E_{12} & \text{ad}(F_1^{(p)})(F_2) &= -F_1^{(p-1)}F_{12} \\ \text{ad}(E_i^{(p)})(F_{12}) &= -F_2E_i^{(p-1)} & \text{ad}(F_i^{(p)})(E_{12}) &= F_i^{(p-1)}E_2 \\ \text{ad}(E_i^{(p)})(H_j) &= 0 & \text{ad}(F_i^{(p)})(H_j) &= 0 \end{aligned}$$

for all  $i, j$ . Thus the derivations  $\text{ad}(E_i^{(p)})$  and  $\text{ad}(F_i^{(p)})$  of  $D_2$ , and the Lie subalgebra of  $\text{Der}(u)$  they generate, preserve  $u$ . Unlike in the rank one case, this does not make  $u$  into an  $\mathfrak{sl}_3(k)$ -module, the problem being that  $\text{Fr} : D_2 \rightarrow u$  is not obviously split as an algebra homomorphism. For example, although  $E_i^{(p)}$  and  $F_i^{(p)}$  generate a subalgebra of  $D$  isomorphic to  $u(\mathfrak{sl}_2(k))$ , the elements  $[E_1^{(p)}, F_1^{(p)}]$  and  $[E_2^{(p)}, F_2^{(p)}]$  do not commute in  $D_2$ : their commutator is a non-zero element of  $u$ . In this way we get a Lie subalgebra of the first Hochschild cohomology  $\text{Der}(u)/\text{IDer}(u)$  isomorphic to  $\mathfrak{sl}_3(k)$ , but because the Levi splitting theorem fails in characteristic  $p$  the surjection from the corresponding subalgebra of  $\text{Der}(u)$  onto  $\mathfrak{sl}_3(k)$  is not guaranteed to split. Some of the other  $\mathfrak{sl}_3(k)$  relations do hold in the hyperalgebra:

**Lemma 4.1.** *In  $D_2$ ,*



- (1)  $[E_i^{(p)}, [E_1^{(p)}, E_2^{(p)}]] = 0 = [F_i^{(p)}, [F_1^{(p)}, F_2^{(p)}]]$  for  $i = 1, 2$ .  
 (2)  $[[E_i^{(p)}, F_i^{(p)}], E_i^{(p)}] = 2E_i^{(p)}$  and  $[[E_i^{(p)}, F_i^{(p)}], F_i^{(p)}] = -2F_i^{(p)}$  for  $i = 1, 2$ .

*Proof.* We prove the  $E$  version of the first identity for  $i = 1$  only. In any universal enveloping algebra, if  $X, Y$  are Lie algebra elements such that  $[X, Y] = Z$  commutes with  $X$  and  $Y$  then

$$[X^{(n)}, Y^{(m)}] = \sum_{i=1}^{\min(m,n)} (-1)^{i+1} X^{(n-i)} Y^{(m-i)} Z^{(i)}.$$

Applying this twice gives

$$[E_1^{(p)}, [E_1^{(p)}, E_2^{(p)}]] = \sum_{i=1}^{p-1} \sum_{j=1}^{p-i} (-1)^{i+j} E_1^{(2p-i-j)} E_2^{(p-i-j)} E_{12}^{(j)} \binom{i+j}{i} \binom{2p-i-j}{p-i}.$$

The product of binomial coefficients vanishes mod  $p$ . The second identity is proved using the same method employed for its quantum analogue in [CK05, Lemma 1].  $\square$

The Lie subalgebra of  $\text{Der}(u)$  generated by the derivations  $\text{ad}(E_i^{(p)})$  and  $\text{ad}(F_i^{(p)})$  preserves the radical of  $u$ :

**Lemma 4.2.** *Let  $d \in D_2, x \in \text{rad}(u)$  and suppose that  $dx - xd \in u$ . Then  $dx - xd \in \text{rad}(u)$ .*

*Proof.* For each simple  $u$ -module  $L$  there is a simple  $D$ -module  $L'$  such that  $L'|_u \cong L$ . The result follows by the characterisation of  $\text{rad}(u)$  as the set of elements annihilating all simple  $u$ -modules.  $\square$

**Lemma 4.3.** *Let  $x$  be a weight-homogeneous element of  $u$  with weight degree  $\pm p\alpha_i$ . Then  $x\varepsilon \in J\varepsilon$  and  $\varepsilon x \in \varepsilon J$ .*

*Proof.* For  $\alpha \in \{\mu, \gamma, \lambda\}$  we have  $u\varepsilon_\alpha/J\varepsilon_\alpha \cong L(\alpha)$  with  $\varepsilon_\alpha + J\varepsilon$  being a highest weight vector. This is killed by any element of the given weight degree.  $\square$

Each derivation  $\delta$  of  $u$  gives rise to a derivation  $\delta_\varepsilon$  of  $\varepsilon u\varepsilon$  by  $\delta_\varepsilon(x) := \varepsilon\delta(x)\varepsilon$  which satisfies  $\delta_\varepsilon(\varepsilon) = 0$ . Let  $e_i$  and  $f_i$  be the derivations of  $\varepsilon u\varepsilon$  arising in this way from  $\text{ad}(E_i^{(p)})$  and  $\text{ad}(F_i^{(p)})$ .

**Lemma 4.4.** *If  $j \in J$  then  $e_i(\varepsilon j\varepsilon) = \varepsilon \text{ad}(E_i^{(p)})(j)\varepsilon$  modulo  $\varepsilon J^2\varepsilon$ .*

*Proof.*  $e_i(\varepsilon j\varepsilon) = \varepsilon \text{ad}(E_i^{(p)})(\varepsilon)j\varepsilon + \varepsilon \text{ad}(E_i^{(p)})(j)\varepsilon + \varepsilon j \text{ad}(E_i^{(p)})(\varepsilon)\varepsilon$ , and the first and last terms on the right lie in  $\varepsilon J^2\varepsilon$  by the previous lemma.  $\square$

$\varepsilon u\varepsilon$ , being Koszul, is isomorphic to the graded algebra arising from its radical filtration  $\varepsilon u\varepsilon \supset \varepsilon J\varepsilon \supset \varepsilon J^2\varepsilon \supset \dots$ . We write  $\text{gr}(\varepsilon u\varepsilon)$  for this graded algebra, so

$$\text{gr}(\varepsilon u\varepsilon) = \frac{\varepsilon u\varepsilon}{\varepsilon J\varepsilon} \oplus \frac{\varepsilon J\varepsilon}{\varepsilon J^2\varepsilon} \oplus \dots$$

A radical-preserving derivation  $\delta$  of  $\varepsilon u\varepsilon$  induces a grading-preserving derivation  $\delta'$  of  $\text{gr}(\varepsilon u\varepsilon)$ . Define a linear map  $\phi : \mathfrak{sl}_3(k) \rightarrow \text{Der}(\text{gr}(\varepsilon u\varepsilon))$  by putting  $\phi(E_i) = e'_i$ ,  $\phi(F_i) = f'_i$ ,  $\phi(H_i) = [e'_i, f'_i]$ ,  $\phi(E_{12}) = [e'_1, e'_2]$ ,  $\phi(F_{12}) = [f'_2, f'_1]$ .

**Lemma 4.5.**  *$\phi$  is a Lie homomorphism.*

*Proof.* We check the relations in  $\mathfrak{sl}_3(k)$  hold for the  $\phi(E_i)$  and  $\phi(F_i)$ . Any relation which holds in  $D_2$  when  $E_i$  is replaced by  $E_i^{(p)}$ ,  $F_i$  by  $F_i^{(p)}$  is satisfied: for example  $[E_1, F_2] = 0$  and

$$[\phi(E_1), \phi(F_2)](\varepsilon j\varepsilon) \equiv \varepsilon [\text{ad}(E_1^{(p)}), \text{ad}(F_2^{(p)})](j)\varepsilon \equiv \varepsilon \text{ad}([E_1^{(p)}, F_2^{(p)}])(j)\varepsilon \equiv 0$$

modulo  $\varepsilon J^2 \varepsilon$ , using Lemma 4.4. Using Lemma 4.2, this leaves only the relations (r1), (r2) and (r3) from Section 1. The first of these follows from the other relations using the Jacobi identity. To get (r2) and (r3) note that  $[[E_1^{(p)}, F_1^{(p)}], E_2^{(p)}] + E_2^{(p)}$  lies in the kernel of  $\text{Fr}$ , and in fact in  $u$ . Thus  $-\text{ad}(E_2^{(p)})$  and  $\text{ad}([E_1^{(p)}, F_1^{(p)}], E_2^{(p)})$  differ by an inner derivation  $\text{ad}(x)$  of  $u$ , where  $|x| = p\alpha_2$ . The relation holds by Lemma 4.3, as do the other relations of this form.  $\square$

If we multiply the weight degrees in  $\mathfrak{sl}_3(k)$  by  $p$  then  $\phi$  respects the weight grading.

**Lemma 4.6.** *As representations of  $\mathfrak{sl}_3(k)$  via  $\phi$ ,*

$$\frac{\varepsilon_\lambda J \varepsilon_\mu}{\varepsilon_\lambda J^2 \varepsilon_\mu} \cong \frac{\varepsilon_\gamma J \varepsilon_\lambda}{\varepsilon_\gamma J^2 \varepsilon_\lambda} \cong L(0, 1) \quad \frac{\varepsilon_\lambda J \varepsilon_\gamma}{\varepsilon_\lambda J^2 \varepsilon_\gamma} \cong \frac{\varepsilon_\mu J \varepsilon_\lambda}{\varepsilon_\mu J^2 \varepsilon_\lambda} \cong L(1, 0).$$

*Proof.* We begin with the case  $\varepsilon_\lambda J \varepsilon_\gamma / \varepsilon_\lambda J^2 \varepsilon_\gamma$ . Consider  $\varepsilon_\lambda F_1^{(p-1-a)} \varepsilon_\gamma$ . This element lies in  $\varepsilon_\lambda J \varepsilon_\gamma \setminus \varepsilon_\lambda J^2 \varepsilon_\gamma$  since under the surjection  $u\varepsilon_\gamma \rightarrow M(\gamma)$  induced by  $\varepsilon_\gamma \mapsto v_\gamma$  it gets sent to an element with non-zero  $M(\gamma)_1$  component by the structure of  $M(\gamma)$  given in Section 2. Applying  $\phi(F_2)$  to  $\varepsilon_\lambda F_1^{(p-1-a)} \varepsilon_\gamma + \varepsilon_\lambda J^2 \varepsilon_\gamma$  sends it to  $\varepsilon_\lambda \text{ad}(F_2^{(p)})(F_1^{(p-1-a)}) \varepsilon_\gamma + \varepsilon_\lambda J^2 \varepsilon_\gamma$ , which is non-zero again by considering the map  $u\varepsilon_\gamma \rightarrow M(\gamma)$ . This proves the claim about non-trivial action.

$(\varepsilon_\lambda J \varepsilon_\gamma) / (\varepsilon_\lambda J^2 \varepsilon_\gamma)$  is three-dimensional, and  $P(\gamma)$  has only  $L(\lambda)$  composition factors shifted to weight degrees  $[p, 0], [-p, p], [0, -p]$ . Since  $\mathfrak{sl}_3(k)$  acts non-trivially and respects this grading, the only possibility is that  $(\varepsilon_\lambda J \varepsilon_\gamma) / (\varepsilon_\lambda J^2 \varepsilon_\gamma)$  is isomorphic to the simple module  $L(1, 0)$ .

The  $e_\lambda J e_\mu$  case can be dealt with similarly, using a homomorphism to  $M(\mu)$ . For  $e_\gamma J e_\lambda$  we need to construct a new  $u$ -module. Let  $M$  be the  $b^+$ -module which is  $M(\gamma)[-p, p] \oplus k_\lambda$  as an  $X$ -graded vector space, where  $k_\lambda = k\langle 1_\lambda \rangle$  is in weight degree  $\lambda$ , with  $b^+$  action on the  $M(\gamma)[-p, p]$  summand defined by restricting the  $u$ -action,  $E_2 \cdot 1_\lambda = 0$  and  $E_1 \cdot 1_\lambda = F_1^{(p-2-a)} v_\gamma$ . We have a short exact sequence

$$0 \rightarrow M(\gamma)[p, -p]|_{b^+} \rightarrow M \rightarrow k_\lambda \rightarrow 0$$

which by Eckmann-Shapiro corresponds to a short exact sequence of  $u$ -modules

$$0 \rightarrow M(\gamma)[p, -p] \rightarrow N \rightarrow M(\lambda) \rightarrow 0$$

Let  $\hat{v}_\lambda$  be a preimage in  $N$  of  $v_\lambda \in M(\lambda)$ , so that there is a surjection  $P(\lambda) \rightarrow N$  sending  $\varepsilon_\lambda$  to  $\hat{v}_\lambda$ . Using the formula at the start of Section 4, we get that  $\text{ad}(E_1^{(p)})(F_1^{(a+1)}) \hat{v}_\lambda$  is a nonzero scalar multiple of  $v_\gamma$ . This can be used to show  $E_1$  acts non-trivially on  $e_\gamma J e_\lambda / e_\gamma J^2 e_\lambda$  as before, and the  $e_\mu J e_\lambda$  case is similar.  $\square$

## 5. PRESENTATION OF THE BASIC ALGEBRA

We now determine a presentation of the basic algebra  $E$ . Let  $\mathbb{k} = E_0 = k\langle e_\lambda, e_\gamma, e_\mu \rangle$  where  $e_\alpha$  is the identity map on  $P(\alpha)$  and zero on the other summands. Let  $V = E_1$ . By Koszulity  $E$  is generated over  $\mathbb{k}$  by  $V$ , and by the previous section  $E$  admits a grading-preserving  $\mathfrak{sl}_3(k)$ -action by derivations such that as  $\mathfrak{sl}_3(k)$ -modules,

$$e_\lambda V e_\gamma \cong e_\mu V e_\lambda \cong L(1, 0) \quad e_\gamma V e_\lambda \cong e_\lambda V e_\mu \cong L(0, 1).$$

Again by Koszulity,  $E$  is quadratic in the sense that  $E \cong T_{\mathbb{k}}(V)/(R)$  where  $T_{\mathbb{k}}$  denotes the tensor algebra and  $R \subseteq V \otimes_{\mathbb{k}} V$ . Because  $\mathfrak{sl}_3(k)$  is acting by grading-preserving derivations,  $R$  is in fact a  $\mathfrak{sl}_3(k)$ -submodule.

The  $\alpha, \beta$  entry of the following table records the  $\mathfrak{sl}_3(k)$ -module structure on  $e_\alpha V \otimes_{\mathbb{k}} V e_\beta$ :

	$\gamma$	$\lambda$	$\mu$
$\gamma$	$L(0,0) \oplus L(1,1)$		$L(0,1) \oplus L(2,0)$
$\lambda$		$L(0,0)^{\oplus 2} \oplus L(1,1)^{\oplus 2}$	
$\mu$	$L(1,0) \oplus L(0,2)$		$L(0,0) \oplus L(1,1)$

Write  $v_{1,0}$ ,  $w_{1,0}$  for highest weight vectors of  $e_\lambda V e_\gamma$  and  $e_\mu V e_\lambda$ , and  $v_{0,1}$ ,  $w_{0,1}$  for highest weight vectors of  $e_\gamma V e_\lambda$  and  $e_\lambda V e_\mu$  respectively.

Corollary 3.6 tells us the dimensions of each  $e_\alpha E_2 e_\beta$ . For example,  $e_\gamma E_2 e_\gamma$  is one-dimensional, and so  $e_\gamma R e_\gamma$  must be a  $\mathfrak{sl}_3(k)$ -submodule of codimension one in  $e_\gamma V \otimes_k V e_\gamma$ , and therefore consists of the  $L(1,1)$  summand from the top row of the previous table, which is generated as a  $\mathfrak{sl}_3(k)$ -module by  $v_{0,1} \otimes v_{1,0}$ . Similarly,  $R$  contains exactly the  $L(2,0)$  summand on the top row of the above table, and the  $L(0,2)$  and  $L(1,1)$  summands on the bottom row.

We still need to identify  $e_\lambda R e_\lambda$ , which must be an eight-dimensional submodule of  $e_\lambda V \otimes_k V e_\lambda$  by Corollary 3.6. It is therefore isomorphic to  $L(1,1)$  as an  $\mathfrak{sl}_3(k)$ -module, and so is generated as an  $\mathfrak{sl}_3(k)$ -module by some element of the form  $r v_{1,0} \otimes v_{0,1} + s w_{0,1} \otimes w_{1,0}$  for  $r, s \in k$ .

Suppose  $r = 0$ . Then in the quadratic dual algebra

$$E^\dagger \cong \text{Ext}_u^*(L(\lambda) \oplus L(\gamma) \oplus L(\mu), L(\lambda) \oplus L(\gamma) \oplus L(\mu))$$

we have  $v_{1,0}^* \cdot v_{0,1}^* = 0$ . Now  $v_{1,0}^*$  corresponds to a weight-graded extension of  $L(\lambda)$  by  $L(\gamma)[p, 0]$  and  $v_{0,1}^*$  corresponds to a weight-graded extension of  $L(\gamma)$  by  $L(\lambda)[0, p]$ . If this product were zero in the Ext-ring, there would be a weight-graded uniserial module with top  $L(\lambda)$ , middle composition factor  $L(\gamma)[-p, 0]$  and socle  $L(\lambda)[-p, -p]$  as in [BC87, Proposition 2.3(a)]. Such a module is necessarily highest weight, hence a quotient of  $M(\lambda)$ , but the structure of  $M(\lambda)$  given in Section 2 shows it has no such quotient. Therefore  $r \neq 0$ , and similarly  $s \neq 0$ . By rescaling  $v_{0,1}$  and the  $\mathfrak{sl}_3(k)$ -submodule it generates, we may assume  $r = s = 1$ . We have proved:

**Theorem 5.1.**  $E \cong T_k(V)/(R)$ , where  $V$  and  $k$  are as above and  $R$  is the  $\mathfrak{sl}_3(k)$ -submodule of  $V \otimes_k V$  generated by  $v_{0,1} \otimes v_{1,0}$ ,  $w_{1,0} \otimes v_{1,0}$ ,  $w_{1,0} \otimes w_{0,1}$ ,  $v_{0,1} \otimes w_{0,1}$ ,  $v_{1,0} \otimes v_{0,1} + w_{0,1} \otimes w_{1,0}$ .

The following table displays  $R$  as a submodule of  $V \otimes_k V$ :

	$\gamma$	$\lambda$	$\mu$
$\gamma$	$L(1,1)$		$L(2,0)$
$\lambda$		Diagonal $L(1,1)$	
$\mu$	$L(0,2)$		$L(1,1)$

## 6. HOCHSCHILD COHOMOLOGY

Any non-semisimple block of  $u(\mathfrak{sl}_2(k))$  is Morita equivalent to a smash product  $\Gamma = kC_2 \ltimes k[x, y]/(x^2, y^2)$  where the cyclic group  $C_2 = \langle g \rangle$  acts by  $gxg = -x$  and  $gyg = -y$ . This can be used to show that the centre  $Z(\Gamma)$  has dimension 3 and  $\text{HH}^1(\Gamma) \cong \mathfrak{gl}_2(k)$  as a Lie algebra.

Since our block algebra is  $K$ -graded, its Hochschild cohomology is bigraded, once by homological degree and once from this  $K$ -grading, referred to as the internal grading. In this section we calculate the centre of our basic algebra and the internal degree zero part of the first Hochschild cohomology, showing that as in the  $\mathfrak{sl}_2$  case, it is isomorphic to the general linear Lie algebra.

Let  $\Lambda = T_k(V)/(R)$ . Let  $v_{-1,0} = F_1 F_2 v_{0,1}$ ,  $v_{1,-1} = F_2 v_{0,1}$ ,  $v_{-1,1} = -F_1 v_{1,0}$ ,  $v_{0,-1} = F_2 F_1 v_{1,0}$  in  $\Lambda$ , and make similar definitions for the  $w$ s. The relations for  $\Lambda$  are then

$$\begin{aligned}
e_\gamma R e_\gamma & \quad v_\alpha v_{-\beta} = 0 \text{ for } \alpha \neq \beta, \\
& \quad v_\alpha v_{-\alpha} = v_\beta v_{-\beta} \text{ for all } \alpha, \beta. \\
e_\mu R e_\mu & \quad w_{-\alpha} w_\beta = 0 \text{ for } \alpha \neq \beta, \\
& \quad w_{-\alpha} w_\alpha = w_{-\beta} w_\beta \text{ for all } \alpha, \beta. \\
e_\mu R e_\gamma & \quad w_{-\alpha} v_{-\beta} + w_{-\beta} v_{-\alpha} = 0 \text{ for all } \alpha, \beta. \\
e_\gamma R e_\mu & \quad v_\alpha w_\beta + v_\beta w_\alpha = 0 \text{ for all } \alpha, \beta. \\
e_\lambda R e_\lambda & \quad v_{-\alpha} v_\beta + w_\beta w_{-\alpha} \text{ for } \beta \neq \alpha, \\
& \quad v_{-\alpha} v_\alpha + w_\alpha w_{-\alpha} = v_{-\beta} v_\beta + w_\beta w_{-\beta} \text{ for all } \alpha, \beta
\end{aligned}$$

where  $\alpha, \beta \in \{(1, -1), (-1, 0), (0, 1)\}$ . Let

$$\begin{aligned}
\mathcal{V} &= \sum v_\alpha v_{-\alpha} & \mathcal{W} &= \sum w_{-\alpha} w_\alpha \\
z_{\gamma\lambda} &= 3v_{0,1}v_{0,-1} + \mathcal{V} - 2\mathcal{W} & z_{\mu\lambda} &= 3w_{0,-1}w_{0,1} + \mathcal{W} - 2\mathcal{V}
\end{aligned}$$

where both sums are over  $\alpha \in \{(-1, 1), (1, 0), (0, -1)\}$ . Let  $z_\alpha$  span the socle of  $\Lambda e_\alpha$  for  $\alpha = \lambda, \mu, \gamma$ .

**Proposition 6.1.** *The centre  $Z(\Lambda)$  has basis  $1, z_\lambda, z_\mu, z_\gamma, z_{\mu\lambda}, z_{\gamma\lambda}$ .*

*Proof.* We first show that the given elements are central. Each of the elements  $z_\alpha$  lie in the centre because they are killed on both sides by any element of  $V$ . Using the relations above:

$v_x$	$v_x \mathcal{V}$	$\mathcal{V} v_x$	$v_x \mathcal{W}$	$\mathcal{W} v_x$
$e_\lambda V e_\gamma$	0	$v_x v_{-x} v_x$	0	$2v_x v_{-x} v_x$
$e_\gamma V e_\lambda$	$v_x v_{-x} v_x$	0	$2v_x v_{-x} v_x$	0

$w_x$	$w_x \mathcal{V}$	$\mathcal{V} w_x$	$w_x \mathcal{W}$	$\mathcal{W} w_x$
$e_\lambda V e_\mu$	$2w_x w_{-x} w_x$	0	$w_x w_{-x} w_x$	0
$e_\mu V e_\lambda$	0	$2w_x w_{-x} w_x$	0	$w_x w_{-x} w_x$

From these it follows easily that  $z_{\gamma\lambda}$  and  $z_{\mu\lambda}$  are central.

Any element of the centre lies in  $e_\lambda \Lambda e_\lambda + e_\gamma \Lambda e_\gamma + e_\mu \Lambda e_\mu$ , since this is the condition to commute with the idempotents  $e_\mu, e_\gamma$  and  $e_\lambda$ . Since it is homogeneous with respect to the  $K$ -grading, it can be non-zero only in degrees 0, 2, 4 by considering the Hilbert series for  $\Lambda$ . Furthermore derivations preserve the centre of an algebra, so  $Z(\Lambda)$  is a  $\mathfrak{sl}_3(k)$ -submodule.

From the Hilbert series,  $e_\alpha \Lambda_2 e_\alpha$  is one-dimensional for  $\alpha = \mu, \gamma$  and  $L(1, 1) \oplus L(0, 0)^{\oplus 2}$  as an  $\mathfrak{sl}_3(k)$ -module for  $\alpha = \gamma$ . The element  $v_{10}v_{01}$  does not commute with  $v_{10}$ , thus the degree two part of the centre is contained in the trivial summands. No nonzero element of the two-dimensional subspace spanned by  $v_{0,1}v_{0,-1}$  and  $w_{1,0}w_{-1,0}$  commutes with both  $v_{01}$  and  $w_{10}$ , so the degree two part of the centre has dimension at most two. It is therefore spanned by  $z_{\gamma\lambda}$  and  $z_{\mu\lambda}$ . Any element of  $\Lambda_4$  is in the span of  $z_\lambda, z_\mu$  and  $z_\gamma$ , so they certainly form a basis of the degree four part of the centre. Lastly a central element of  $\Lambda_0$  must be a scalar multiple of the identity, so we are done.  $\square$

We already know some non-inner  $K$ -grading-preserving derivations of  $\Lambda$ , namely those induced by the action of  $\mathfrak{sl}_3(k)$  and the grading derivation  $\Delta$  defined by  $\Delta(l) = nl$  for  $l \in \Lambda_n$ .

**Proposition 6.2.** *The internal degree zero part of  $\mathrm{HH}^1(\Lambda)$  is isomorphic to  $\mathfrak{gl}_3(k)$  and is spanned by the image of  $\Delta$  together with the images of the derivations arising from the  $\mathfrak{sl}_3(k)$ -action.*

*Proof.* Let  $\delta$  be a non-inner derivation of  $\Lambda$  which preserves the  $K$ -grading. Since  $\delta$  preserves  $\Lambda_0 = \langle e_\lambda, e_\gamma, e_\mu \rangle$  and the  $e_\alpha$  are idempotent, it must act as zero on  $\Lambda_0$ . Therefore  $\delta$  is determined by its action on  $\Lambda_1$ , and must send  $e_\alpha \Lambda_1 e_\beta$  to  $e_\alpha \Lambda e_\beta$  for each  $\alpha, \beta$ .

So  $\delta$  is determined by the four linear maps  $\delta_{\alpha\beta} \in \text{End}_k(e_\alpha \Lambda_1 e_\beta)$ , for  $(\alpha, \beta) = (\gamma, \lambda), (\lambda, \gamma), (\mu, \lambda), (\lambda, \mu)$ . Any such maps extend uniquely to a derivation of  $T_k(V)$ . The condition for  $\delta$  to be a derivation is that these maps, when extended to a derivation on  $T_k(V)$ , must preserve  $R \subseteq V \otimes_k V$ . Identify  $e_\gamma V e_\lambda$  and  $e_\lambda V e_\mu$  with  $L(0, 1)$ , and  $e_\lambda V e_\gamma$  and  $e_\mu V e_\lambda$  with  $L(0, 1)^*$ . Then  $e_\mu R e_\gamma$  is the symmetric square of  $L(0, 1)$ , and if  $a$  and  $b$  are linear endomorphisms of  $L(0, 1)$  then  $1 \otimes a + b \otimes 1$  preserves the symmetric square of  $L(0, 1)$  if and only if  $a$  and  $b$  differ by a scalar multiple of the identity. Thus  $\delta_{\mu\lambda} - \delta_{\lambda\gamma}$  and  $\delta_{\gamma\lambda} - \delta_{\lambda\mu}$  are scalar multiples of the identity.

Write  $\delta_{\mu\lambda} = rI + x$  and  $\delta_{\lambda\mu} = sI + y$  where  $I$  is the identity,  $r, s \in k$  and  $x$  and  $y$  are the linear endomorphisms of  $L(0, 1)$  and  $L(0, 1)^*$  induced by the action of elements  $X, Y \in \mathfrak{sl}_3(k)$ . Note that under our identifications,  $e_\mu R e_\mu$  is the kernel of the evaluation map  $\nu : L(0, 1) \otimes L(0, 1)^* \rightarrow k$ . We claim that the endomorphism

$$(\delta_{\mu\lambda} - rI) \otimes 1 + 1 \otimes (\delta_{\lambda\mu} - sI) = x \otimes 1 + 1 \otimes y$$

has image contained in  $\ker \nu$ . Note  $L(0, 1) \otimes L(0, 1)^* = \ker \nu \oplus L(0, 0)$  where the trivial summand is spanned by  $v = \sum v_{ij} \otimes v_{ij}^*$ , and that  $\ker \nu$  belongs to  $R$  so is preserved by this endomorphism since it differs from the action of  $\delta$  by a scalar multiple of the identity. Both  $x \otimes 1$  and  $1 \otimes y$  act on  $v$  by scalar multiplication by their trace, which is zero, proving the claim.

This shows that for any  $u \in L(0, 1)$  and  $f \in L(0, 1)^*$ ,

$$f(x)(u) + (y(f))(u) = 0$$

that is,  $(Y \cdot f)(u) = f(-X \cdot u)$  for all  $f, u$ . But by definition of the action on the dual space,  $(Y \cdot f)(u) = f(-Y \cdot u)$ . It follows  $X = Y$ . We now have

$$\begin{aligned} \delta_{\mu\lambda} &= rI + \rho(X) & \delta_{\lambda\gamma} &= r'I + \rho(X) \\ \delta_{\lambda\mu} &= sI + \rho'(X) & \delta_{\gamma\lambda} &= s'I + \rho'(X) \end{aligned}$$

for some  $X \in \mathfrak{sl}_3(k)$  and  $r, r', s, s' \in k$  where  $\rho$  and  $\rho'$  are the representations corresponding to  $L(0, 1)$  and its dual. The only part of  $R$  we are yet to consider is  $e_\lambda R e_\lambda$ , but  $\delta$  preserves this if and only if  $r + s = r' + s'$ . Adding

$$((r + s)/2 + r') \text{ad}(e_\gamma) + ((s - r)/2) \text{ad}(e_\mu)$$

shows  $\delta$  is  $((r + s)/2)\Delta$  plus the action of an element of  $\mathfrak{sl}_3(k)$ .

The inner derivations of degree zero are spanned by the  $\text{ad}(e_\alpha)$ s which are linearly independent of  $\Delta$  and the  $\mathfrak{sl}_3(k)$ -action. Any linear dependence could only involve the  $H_i$  and  $\Delta$  action because the  $\text{ad}(e_\alpha)$  preserve the weight grading.  $\Delta$  and the  $\text{ad}(e_\alpha)$  act as a scalar on  $e_\lambda V e_\gamma$ , but no nonzero linear combination of the  $H_i$  does, so the  $H_i$  cannot be involved. Finally  $\Delta$  is not a linear combination of  $\text{ad}(e_\gamma)$  and  $\text{ad}(e_\mu)$  since it acts by the same scalar on  $e_\gamma V e_\lambda$  as it does on  $e_\lambda V e_\mu$ .  $\square$

## 7. VERMA MODULES IN THIS BLOCK ARE KOSZUL

Recall that a graded module  $M$  over a Koszul algebra  $\Gamma$  is called Koszul if it has a graded projective resolution  $P^* \rightarrow M$  such that  $P^i = \Gamma P_i^i$ . Such a projective resolution is called linear. Equivalently  $M$  is Koszul if  $\text{Ext}_\Gamma^{i,j}(M, (\Gamma/\Gamma_{>0}))$  is zero unless  $i = j$ , where  $i$  is the homological degree and  $j$  the internal degree arising from the  $\mathbb{Z}_{\geq 0}$ -grading on  $\Gamma$ . Koszulity implies  $M \cong \Gamma \otimes_k M_0/H$  where  $H \subseteq \Gamma_1 \otimes M_0$ . In this section we show that the Verma modules  $M(\lambda)$ ,  $M(\gamma)$  and  $M(\mu)$  are Koszul.

$\Lambda$  has an  $X$ -grading obtained by putting the  $e_\alpha$  in degree  $(0, 0)$  and the  $v_{i,j}$  and  $w_{i,j}$  in degree  $(pi, pj)$ . We refer to this as the weight grading of  $\Lambda$ .

**Lemma 7.1.** *Under the Morita equivalence between our block of  $u$  and  $\Lambda$ , the Verma modules  $M(\lambda), M(\gamma), M(\mu)$  correspond to*

$$M_\lambda = \frac{\Lambda e_\lambda}{\Lambda v_{0,1} + \Lambda v_{1,-1} + \Lambda w_{-1,1} + \Lambda w_{1,0}}, M_\gamma = \frac{\Lambda e_\gamma}{\Lambda v_{1,0}}, M_\mu = \frac{\Lambda e_\mu}{\Lambda w_{0,1}}$$

respectively.

*Proof.* Let  $S_\alpha$  be the weight-graded simple  $\Lambda$ -module  $\Lambda e_\alpha / J(\Lambda) e_\alpha$ . The Morita correspondent of  $M(\gamma)$  is a quotient  $\Lambda e_\gamma / I_\gamma$  with weight-graded composition factors  $S_\gamma, S_\lambda[-p, p], S_\lambda[0, -p], S_\mu[-p, 0]$ . So  $I_\gamma$  must contain  $v_{1,0}$ , but  $\Lambda e_\gamma / \Lambda v_{1,0}$  already has only four composition factors, so it must be the Morita correspondent of  $M(\gamma)$ . The other correspondences follow similarly.  $\square$

$M_\gamma$  has a basis consisting of the images of  $e_\gamma, v_{-1,1}, v_{0,-1}$  and  $w_{-1,1}v_{0,-1} = -w_{0,-1}v_{-1,1}$ ,  $M_\mu$  has a basis consisting of the images of  $e_\mu, w_{-1,0}, w_{1,-1}$  and  $v_{1,-1}w_{-1,0} = -v_{-1,0}w_{1,-1}$ , and  $M_\lambda$  has a basis consisting of the images of  $e_\lambda, w_{0,-1}, v_{-1,0}, v_{0,-1}v_{-1,0} = -w_{-1,0}w_{0,-1}$ .

Put  $I = \Lambda v_{0,1} + \Lambda w_{1,0}$  and  $M = \Lambda e_\lambda / I$ , regarded as a  $K$ -graded module with  $e_\lambda + I$  in degree zero.

**Lemma 7.2.** *There is an exact sequence of  $K$ -graded modules*

$$(3) \quad 0 \rightarrow M_\gamma[1] \oplus M_\mu[1] \xrightarrow{i} M \xrightarrow{\pi} M_\lambda \rightarrow 0$$

where  $i(e_\gamma + \Lambda v_{1,0}) = v_{1,-1} + I$ ,  $i(e_\mu + \Lambda w_{0,1}) = w_{-1,1} + I$  and  $\pi$  is the quotient map.

*Proof.*  $i$  is well-defined because  $w_{0,1}w_{-1,1} = -v_{-1,1}v_{0,1} \in I$  and  $v_{1,0}v_{1,-1} = -w_{1,-1}w_{1,0} \in I$ , and  $\text{im } i$  is the kernel of the quotient map  $M \rightarrow M_\lambda$ . We only need to show  $i$  is injective, and it is enough to prove  $i$  is injective on the socle  $S_\mu \oplus S_\gamma$  of  $M_\gamma \oplus M_\mu$ . Since the two simple summands are non-isomorphic (or by considering the weight degree), it is enough simply to show that  $i$  is nonzero on  $\text{soc } M_\gamma$  and on  $\text{soc } M_\mu$ .

$i(\text{soc } M_\mu)$  is spanned by the image of  $v_{1,-1}w_{-1,0}w_{-1,1}$  in  $M$ , which is homogeneous of  $K$ -degree 3 and weight degree  $[-p, 0]$ . We want to show this element does not lie in  $I = \Lambda v_{0,1} + \Lambda w_{1,0}$ . We claim  $\Lambda v_{0,1}$  has no nonzero element in this  $K$ - and weight-degree. To get such an element, we would have to multiply  $v_{0,1}$  by two of the generators  $v_{i,j}, w_{i,j}$ , and the first has to be  $v_{0,-1}$  as all other products are either zero or too far from  $[-p, 0]$  (it helps to look at Figure 3 to see this). The second must then be  $v_{-1,0}$  if we are to end up at weight degree  $[-p, 0]$ , but  $v_{-1,0}v_{0,-1}v_{0,1} = 0$  because  $v_{-1,0}v_{0,-1} = 0$ . Similarly,  $\Lambda w_{1,0}$  has no nonzero  $K$ -degree 3 weight degree  $[-p, 0]$  element: the only possible product is  $v_{-1,0}w_{-1,0}w_{1,0}$  as before, but  $v_{-1,0}w_{-1,0} = 0$ .

A similar argument shows that  $i$  is injective on  $\text{soc } M_\gamma$ , completing the proof.  $\square$

Let  $\Omega(M) = (\Lambda e_\mu \oplus \Lambda e_\gamma) / \Lambda(w_{0,1} + v_{1,0})$ .

**Lemma 7.3.** *There are exact sequences of  $K$ -graded modules*

$$(4) \quad 0 \rightarrow M_\lambda[1] \xrightarrow{\psi} \Omega(M) \xrightarrow{\phi} M_\gamma \oplus M_\mu \rightarrow 0$$

$$(5) \quad 0 \rightarrow M[1] \xrightarrow{i} \Lambda e_\mu[1] \oplus \Lambda e_\gamma[1] \xrightarrow{j} \Lambda e_\lambda \rightarrow M \rightarrow 0$$

*Proof.*  $\phi((a, b) + \Lambda(w_{0,1} + v_{1,0})) = (a + \Lambda w_{0,1}, b + \Lambda v_{1,0})$  is well-defined and surjective, with kernel

$$\ker \phi = \frac{\Lambda v_{1,0} + \Lambda w_{0,1}}{\Lambda(v_{1,0} + w_{0,1})}$$

generated by the image of  $v_{1,0}$ . Therefore  $M_\lambda$  surjects onto  $\ker \phi$  via the map  $\psi$  induced by  $e_\lambda \mapsto v_{1,0} + \Lambda(v_{1,0} + w_{0,1})$ . We only need  $\psi$  to be injective. If we

show that  $\dim \Omega(M) \geq 12$  then  $\ker \phi$  has dimension at least four, but since it is a quotient of  $M_\lambda$  it will be exactly four and hence  $\psi$  will have been shown to be injective.

Letting  $\Lambda e_\lambda \rightarrow M$  be the quotient map and defining  $j$  by  $j(e_\mu) = w_{1,0}$ ,  $j(e_\gamma) = v_{0,1}$ , we get an exact sequence

$$\Lambda e_\gamma[1] \oplus \Lambda e_\mu[1] \xrightarrow{j} \Lambda e_\lambda \rightarrow M \rightarrow 0.$$

$v_{1,0} + w_{0,1} \in \ker j$  so  $j$  factors through  $\Omega(M)[1]$ , showing that  $\dim \Omega(M) \geq \dim \operatorname{im} j = \dim \Lambda e_\lambda - \dim M = 12$ . This completes the proof of the exactness of (4), so  $\dim \Omega(M) = 12$ .

$\dim \operatorname{im} j = 12$ , so  $\dim \ker j = 12$ , and so  $\Lambda(v_{1,0} + w_{0,1})$ , which has dimension 12 as  $\Omega(M)$  does, must be all of  $\ker j$ . Therefore  $M[1]$  surjects onto  $\ker j$  by  $i(e_\lambda + I) = v_{1,0} + w_{0,1}$  (this is well-defined). Since  $\dim M = 12$ ,  $i$  must be injective. This gives exactness of (5).  $\square$

**Corollary 7.4.**  *$M$  and  $\Omega(M)$  are Koszul.*

*Proof.* Splicing shifted copies of (5) produces a linear projective resolution of  $M$ . Koszulity of  $\Omega(M)$  follows by truncating and shifting the same resolution.  $\square$

**Theorem 7.5.**  *$M_\lambda, M_\gamma$  and  $M_\mu$  are Koszul.*

*Proof.* Let  $S = S_\mu \oplus S_\lambda \oplus S_\gamma$ . We will show that for every  $n$ , the graded vector spaces  $\operatorname{Ext}_\Lambda^{n,m}(M_\lambda, S)$  and

$$\operatorname{Ext}_\Lambda^{n,m}(M_\gamma \oplus M_\mu, S) \cong \operatorname{Ext}_\Lambda^{n,m}(M_\gamma, S) \oplus \operatorname{Ext}_\Lambda^{n,m}(M_\mu, S)$$

are zero unless  $m = n$ . This is certainly true for  $n = 0$ .

Consider the long exact sequence obtained by applying  $\operatorname{Hom}_\Lambda(-, S)$  to (3). Let  $\omega$  be the connecting homomorphism: it corresponds to Yoneda multiplication by the short exact sequence (3) and is therefore homogeneous of  $K$ -degree one. For each  $n$  we get a short exact sequence

$$0 \rightarrow \frac{\operatorname{Ext}_\Lambda^n(M_\gamma \oplus M_\mu, S)}{\ker \omega} \rightarrow \operatorname{Ext}_\Lambda^{n+1}(M_\lambda, S) \xrightarrow{\pi^*} \operatorname{im} \pi^* \rightarrow 0$$

in which the first map, induced by  $\omega$ , increases  $K$ -degree by one, and the second preserves  $K$ -degree.  $M$  is Koszul, so  $\operatorname{im} \pi^* \subseteq \operatorname{Ext}_\Lambda^{n+1}(M, S)$  is zero outside  $K$ -degree  $n + 1$ . It follows that if  $\operatorname{Ext}_\Lambda^n(M_\gamma \oplus M_\mu, S)$  is zero outside  $K$ -degree  $n$  then  $\operatorname{Ext}_\Lambda^{n+1}(M_\lambda, S)$  is zero outside  $K$ -degree  $n + 1$ .

Similar short exact sequences arising from the long exact sequence obtained by applying  $\operatorname{Hom}_\Lambda(-, S)$  to (4) show that if  $\operatorname{Ext}_\Lambda^n(M_\lambda, S)$  is zero outside  $K$ -degree  $n$  then  $\operatorname{Ext}_\Lambda^{n+1}(M_\gamma \oplus M_\mu, S)$  is zero outside  $K$ -degree  $n + 1$ . Since the result holds for  $n = 0$ , this completes the proof.  $\square$

## REFERENCES

- [AJS94] H. H. Andersen, J. C. Jantzen, and W. Soergel, *Representations of quantum groups at a  $p$ th root of unity and of semisimple groups in characteristic  $p$ : independence of  $p$* , *Astérisque* (1994), no. 220, 321. MR 1272539 (95j:20036)
- [BC87] David J. Benson and Jon F. Carlson, *Diagrammatic methods for modular representations and cohomology*, *Comm. Algebra* **15** (1987), no. 1-2, 53–121. MR 876974 (87m:20032)
- [Ben98] D. J. Benson, *Representations and cohomology. I*, second ed., *Cambridge Studies in Advanced Mathematics*, vol. 30, Cambridge University Press, Cambridge, 1998, Basic representation theory of finite groups and associative algebras. MR 1644252 (99f:20001a)
- [BGS96] Alexander Beilinson, Victor Ginzburg, and Wolfgang Soergel, *Koszul duality patterns in representation theory*, *J. Amer. Math. Soc.* **9** (1996), no. 2, 473–527. MR 1322847 (96k:17010)
- [CK05] W. Chin and L. Krop, *Quantized hyperalgebras of rank 1*, *Israel J. Math.* **145** (2005), 193–219. MR 2154726 (2006d:17012)

- [GG82] Robert Gordon and Edward L. Green, *Representation theory of graded Artin algebras*, J. Algebra **76** (1982), no. 1, 138–152. MR 659213 (83m:16028b)
- [HN91] Randall R. Holmes and Daniel K. Nakano, *Brauer-type reciprocity for a class of graded associative algebras*, J. Algebra **144** (1991), no. 1, 117–126. MR 1136900 (93b:16015)
- [Hum78] James E. Humphreys, *Introduction to Lie algebras and representation theory*, Graduate Texts in Mathematics, vol. 9, Springer-Verlag, New York, 1978, Second printing, revised. MR 499562 (81b:17007)
- [Irv86] Ronald S. Irving, *The structure of certain highest weight modules for  $SL_3$* , J. Algebra **99** (1986), no. 2, 438–457. MR 837554 (87h:17008)
- [Jan03] Jens Carsten Jantzen, *Representations of algebraic groups*, second ed., Mathematical Surveys and Monographs, vol. 107, American Mathematical Society, Providence, RI, 2003. MR 2015057 (2004h:20061)
- [Ric10] Simon Riche, *Koszul duality and modular representations of semisimple Lie algebras*, Duke Math. J. **154** (2010), no. 1, 31–134. MR 2668554 (2011m:17021)
- [Xi99] Nanhua Xi, *Maximal and primitive elements in Weyl modules for type  $A_2$* , J. Algebra **215** (1999), no. 2, 735–756. MR 1686213 (2001e:17028)

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